Leverage Aversion, Efficient Frontiers, and the Efficient Region

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We have proposed that portfolio theory and mean–variance optimization (Markowitz [1952, 1959]) be augmented to incorporate investor aversion to leverage, and we have shown that mean–variance-leverage optimization creates a three-dimensional surface of optimal portfolios (Jacobs and Levy [2012, 2013]).

Conventional mean–variance optimization determines optimal security weights by considering a portfolio's expected return and variance of portfolio return. To the extent that leverage increases a portfolio's volatility (the square root of variance), mean–variance optimization captures some of the risk associated with leverage.[1] But it does not consider other components of risk that are unique to using leverage. These include the risks and costs of margin calls, which can force borrowers to liquidate securities at adverse prices due to illiquidity, losses exceeding the capital invested, and the possibility of bankruptcy.[2]

Mean–variance analysis results in optimal unleveraged (long-only) portfolios for investors who are not able to tolerate any leverage. But for investors who use leverage, mean–variance analysis can result in optimal portfolios that are highly leveraged. This is because mean–variance optimization implicitly assumes that the investor has an infinite tolerance for, or (stated differently) no aversion to, the unique risks of leverage.

In practice, however, investors *are* leverage averse. For example, if offered a choice between a portfolio with a particular expected return and variance *without* leverage and another portfolio that offers the same expected return and variance *with* leverage, most investors would prefer the portfolio without leverage. The conventional mean–variance utility function cannot distinguish between these two portfolios, because it does not account for an important aspect of investors' behavior: investors' aversion to the unique risks of leverage.

Investors who use leverage usually limit it, choosing a leverage level and imposing it on the portfolio with a constraint of the type described in Markowitz [1959]. Jacobs and Levy [2012] suggested determining the optimal leverage level by using a utility function that includes an explicit leverage-tolerance term, in addition to the traditional volatility-tolerance term.[3] That article provided one way to specify the leverage-tolerance term and illustrated optimal portfolio leverage levels when the utility function includes both volatility and leverage aversion.

In this article, we provide an alternative specification of the leverage-tolerance term, which may better capture the unique risks of leverage. We introduce mean–variance-leverage efficient frontiers, display them in the familiar two-dimensional form of mean–variance analysis, and compare them with conventional mean–variance efficient frontiers. We also develop the concept of a mean–variance-leverage efficient region, bounded by a range of volatility tolerance and leverage tolerance. An analysis of the mean–variance-leverage efficient frontiers and the efficient region shows that leverage aversion can have a large effect on portfolio choice.

SPECIFYING THE LEVERAGE-AVERSION TERM

The leverage-aversion term that augments a mean-variance utility function can be specified in different ways. Jacobs and Levy [2012] suggested the following:

$$U = \alpha_P - \frac{1}{2\tau_V}\sigma_P^2 - \frac{1}{2\tau_L} c\,\Lambda^2 \tag{1}$$

where α_P is the portfolio's expected active return relative to benchmark, σ_P^2 is the variance of the portfolio's active return, Λ is the portfolio's leverage, and c is a constant defined in Equation (3).[4] With this specification, risk tolerance essentially changes from a one-dimensional attribute (as in mean–variance optimization) to a two-dimensional attribute, with the first dimension being the traditional risk tolerance, renamed as volatility tolerance τ_V, and the second dimension being leverage tolerance, τ_L. We used a squared term for leverage so that both risk components would have similar functional forms. We define leverage as

$$\Lambda = \sum_{i=1}^{N} |h_i| - 1 \tag{2}$$

where h_i is the portfolio holding weight of security i for each of the N securities in the selection universe.[5]

To investigate this utility function, we determined illustrative ranges for the tolerances. As one reference point, a value of $\tau_V = 0$ corresponds to an investor who is completely intolerant of active volatility risk. Such an investor would choose an index fund. As another reference point, a value of $\tau_V \approx 1$ causes quadratic utility of return to be equivalent to log-utility of wealth, a utility function often used in the finance literature (Levy and Markowitz [1979]). Thus, we chose $\tau_V \in [0,2]$. For illustrative purposes, we chose τ_L to span the same range as τ_V.

We selected a constant c that would result in the two risk terms (volatility risk, σ_P^2, and leverage risk, $c\Lambda^2$) having similar orders of magnitude. In particular, c was chosen to be the cross-sectional average of the variances of the securities' active returns. That is,

$$c = \frac{1}{N}\sum_{i=1}^{N} \omega_i^2 \tag{3}$$

where ω_i^2 is the variance of the active return of security i. Because portfolios in practice generally have leverage levels ranging from 0 to about 2 (very highly leveraged portfolios are relatively few in number, but can be large in asset size), the product $c\Lambda^2$ should be of a similar order of magnitude to σ_P^2, so that similar values of τ_V and τ_L lead to similar levels of disutility.

Using the constant c to specify the leverage-tolerance term has a certain intuitive appeal. In addition to resulting in similar orders of magnitude for the volatility and leverage terms, using active returns in computing c is congruent with using active returns in computing a portfolio's expected active return and variance. Moreover, from an implementation perspective, the use of a constant means that the utility function can, if desired, be restated as a quadratic optimization problem, which is advantageous because quadratic solvers are readily available.

However, the unique risks of leverage relate more to a portfolio's total volatility than to the volatility of its active returns. That is, the risk that portfolio losses will trigger a margin call or exceed the capital invested depends on the portfolio's total volatility. Furthermore, this leverage dimension of risk will not be constant; rather, it will vary across different portfolios having different volatilities.

SPECIFICATION OF THE LEVERAGE-AVERSION TERM USING PORTFOLIO TOTAL VOLATILITY

Here we introduce another possible specification of an augmented mean–variance utility function that includes a leverage-aversion term:

$$U = \alpha_P - \frac{1}{2\tau_V}\sigma_P^2 - \frac{1}{2\tau_L}\sigma_T^2\Lambda^2 \tag{4}$$

where σ_T^2 is the variance of the leveraged portfolio's total return. This leverage-aversion term assumes that the risks of leverage rise with the product of the variance of the leveraged portfolio's total return and the square of the portfolio's leverage. This specification may better capture the portfolio's risk of margin calls and forced liquidations.

If α_i is the expected active return of security i, b_i is the weight of security i in the benchmark, x_i is the active weight of security i (and by definition $x_i = h_i - b_i$), σ_{ij} is the covariance between the active returns of securities i and j, and q_{ij} is the covariance between the total returns of securities i and j, then Equation (4) can be written as

$$U = \sum_{i=1}^{N} \alpha_i x_i - \frac{1}{2\tau_V} \sum_{i=1}^{N} \sum_{j=1}^{N} x_i \sigma_{ij} x_j - \frac{1}{2\tau_L} \left(\sum_{i=1}^{N} \sum_{j=1}^{N} h_i q_{ij} h_j \right) \Lambda^2 \tag{5}$$

Using Equation (2), and because $h_i = b_i + x_i$, Equation (5) becomes:

$$U = \sum_{i=1}^{N} \alpha_i x_i - \frac{1}{2\tau_V} \sum_{i=1}^{N} \sum_{j=1}^{N} x_i \sigma_{ij} x_j - \frac{1}{2\tau_L} \left(\sum_{i=1}^{N} \sum_{j=1}^{N} \left(b_i + x_i\right) q_{ij} \left(b_j + x_j\right) \right) \left(\sum_{i=1}^{N} \left|b_i + x_i\right| - 1 \right)^2 \tag{6}$$

Equation (6) is the utility function to be maximized. In practice, the utility function in Equation (6) is more difficult to optimize than that in Equation (1), because the leverage-risk term requires powers up to and including the fourth order in the x_i terms. We will show a method to solve for optimal portfolios using this utility function.

OPTIMAL PORTFOLIOS WITH LEVERAGE AVERSION BASED ON PORTFOLIO TOTAL VOLATILITY

To examine the effects of leverage aversion using this new specification, we used the enhanced active equity (EAE) portfolio structure, as in Jacobs and Levy [2012]. An EAE portfolio has 100% exposure to an underlying market benchmark, while permitting short sales equal to some percentage of capital and use of the short-sale proceeds to buy additional long positions. For expository purposes, we assume the strategy is self-financing and entails no financing costs.[6] An enhanced active 130–30 portfolio, for instance, has leverage of 60% and an enhancement of 30%.

We found EAE portfolios that maximize the utility function represented by Equation (6) for a range of volatility and leverage tolerance pairs (τ_V, τ_L), subject to standard constraints. The standard constraint set for an EAE portfolio is

$$\sum_{i=1}^{N} x_i = 0 \tag{7}$$

and

$$\sum_{i=1}^{N} x_i \beta_i = 0 \tag{8}$$

Equation (7) says the sum of security active underweights relative to benchmark (including short positions) equals the sum of security active overweights—the full investment constraint. Equation (8) says that the sum of the products of security active weights and security betas equals zero; that is, the net (long–short) portfolio beta equals the benchmark beta. In addition to these standard constraints, we constrained each security's active weight to be between −10% and +10%.

We maximized the utility function in Equation (6) by using a fixed-point iteration. To explain this procedure, we rewrite Equation (6) as the following set of two equations:

$$U = \sum_{i=1}^{N} \alpha_i x_i - \frac{1}{2\tau_V} \sum_{i=1}^{N} \sum_{j=1}^{N} x_i \sigma_{ij} x_j - \frac{1}{2\tau_L} \sigma_T^2 \left(\sum_{i=1}^{N} \left|b_i + x_i\right| - 1 \right)^2$$

$$\sigma_T^2 = \sum_{i=1}^{N} \sum_{j=1}^{N} \left(b_i + x_i\right) q_{ij} \left(b_j + x_j\right) \tag{9}$$

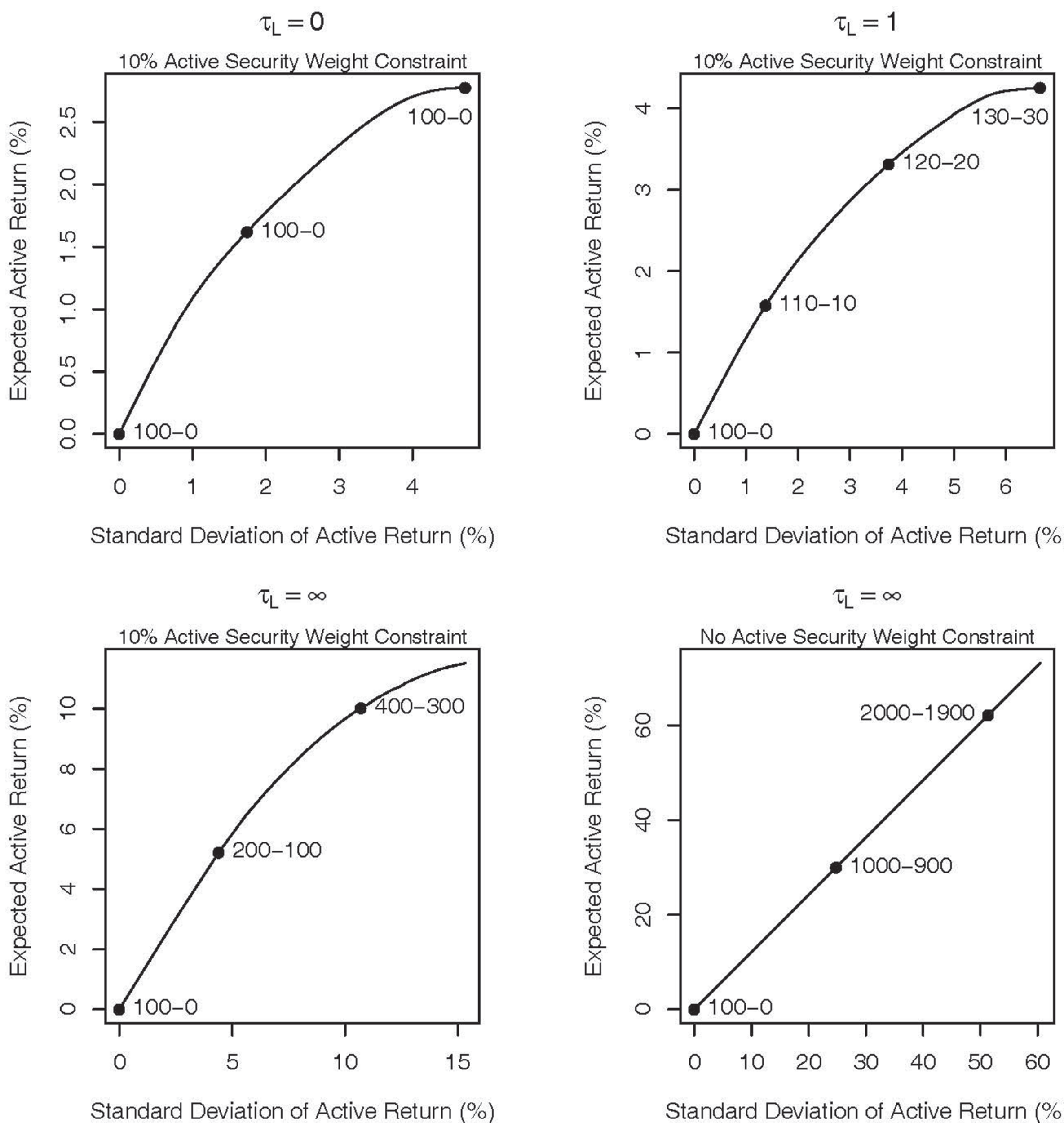
We chose an initial estimate of σ_T^2 and used this as a constant to maximize the utility function in Equation Set (9). This maximization provided estimates of the x_is, which we used to compute a new estimate of σ_T^2 using the second equation in Equation Set (9). With the new estimate of σ_T^2, we repeated the optimization to find new estimates of the x_is. We repeated this iteration until successive estimates of σ_T^2 differed by a *de minimis* amount.

Using the same data (for stocks in the S&P 100 Index) and estimation procedures used in Jacobs and Levy [2012], and the same range of leverage and volatility tolerances, we derived the enhancement surface for the optimal levels of portfolio leverage using the new specification of leverage aversion.[7] The optimal levels of enhancement were slightly higher than, but substantially similar to, those of the earlier specification. The appendix explains the reasons for the small differences in the optimal levels of enhancement between the two specifications.

EFFICIENT FRONTIERS WITH AND WITHOUT LEVERAGE AVERSION

Exhibit 1 illustrates, in a familiar two-dimensional volatility risk–return framework, how considering leverage aversion can affect the investor's choice of an optimal portfolio. We plot efficient frontiers (the set of optimal

EXHIBIT 1
Optimal Leverage for Various Leverage-Tolerance (τ_L) Cases



portfolios) for four cases. We compute the frontiers as discussed in the previous section, with the fourth frontier computed without the 10% constraint on active security weights. The portfolios on these frontiers offer the highest expected active return at each given level of volatility (whether measured as variance or as standard deviation of active return). We map out the frontier in each separate chart by varying the level of volatility tolerance from 0 to 2 while holding the level of leverage tolerance constant.

In all the cases illustrated in Exhibit 1, the efficient frontier begins at the origin, which corresponds to the optimal portfolio when volatility tolerance is 0. In such a situation, the investor cannot tolerate any active volatility, so the optimal portfolio is an index fund, which provides zero standard deviation of active return and thus zero expected active return. As the investor's volatility tolerance increases, the optimal portfolio moves out along the efficient frontier.

The first panel of the exhibit illustrates the efficient frontier when leverage tolerance is 0, meaning the investor is unwilling to use leverage and hence holds a long-only portfolio. As the investor's tolerance for volatility increases, the optimal portfolio moves out along the frontier, taking on higher levels of standard deviation of active return in order to earn higher levels of expected active return. These portfolios take more concentrated positions in securities with higher expected returns as volatility tolerance increases.

We can derive the efficient frontier when leverage (including shorting) is not used from either a conventional mean–variance optimization with a zero leverage constraint or from a mean–variance-leverage optimization with zero tolerance for leverage. As noted on the exhibit, every portfolio along the frontier is a "100–0" portfolio, meaning it is invested 100% long, with no short positions.

The second panel illustrates the efficient frontier when the investor's leverage tolerance is 1. It is derived from a mean–variance-leverage optimization, where leverage entails a disutility, as specified in Equation (4). Again, the investor with no tolerance for volatility risk would hold the index fund located at the origin. But as investor tolerance for volatility increases, the optimal portfolio moves out along the efficient frontier, achieving higher levels of expected return with higher levels of volatility.

As the plot indicates, increasing leverage accompanies increasing volatility. The optimal portfolio ranges from a 100–0 long-only portfolio to a 130–30 enhanced active portfolio. For the investor with a leverage tolerance of 1, any of these portfolios can be optimal, depending on volatility tolerance. Investors can achieve higher risk–return portfolios with less concentration of positions when leverage is allowed than when leverage is not allowed.

The third panel illustrates the efficient frontier for an investor with infinite leverage tolerance. As discussed earlier, mean–variance optimization implicitly assumes investors have an infinite tolerance for (or no aversion to) the unique risks of leverage; it thus provides the same result as mean–variance-leverage optimization with infinite leverage tolerance. In this case, as the investor's volatility tolerance increases, the optimal portfolio goes from zero leverage to enhanced active portfolios of 200–100 to 400–300, and so on. For the investor with infinite leverage tolerance, the unique risks of leverage do not give rise to any disutility, so this investor takes on much more leverage than the investors in the prior examples and can achieve a higher expected return at any given level of standard deviation of return, albeit with increasing leverage risk.

The last panel is identical to the third, except that it removes the 10% constraint on individual security active weights. Because there is no disutility to leverage and no constraint on individual position sizes, the optimal portfolios all hold the same proportionate active security weights but apply increasing levels of leverage as volatility tolerance increases.

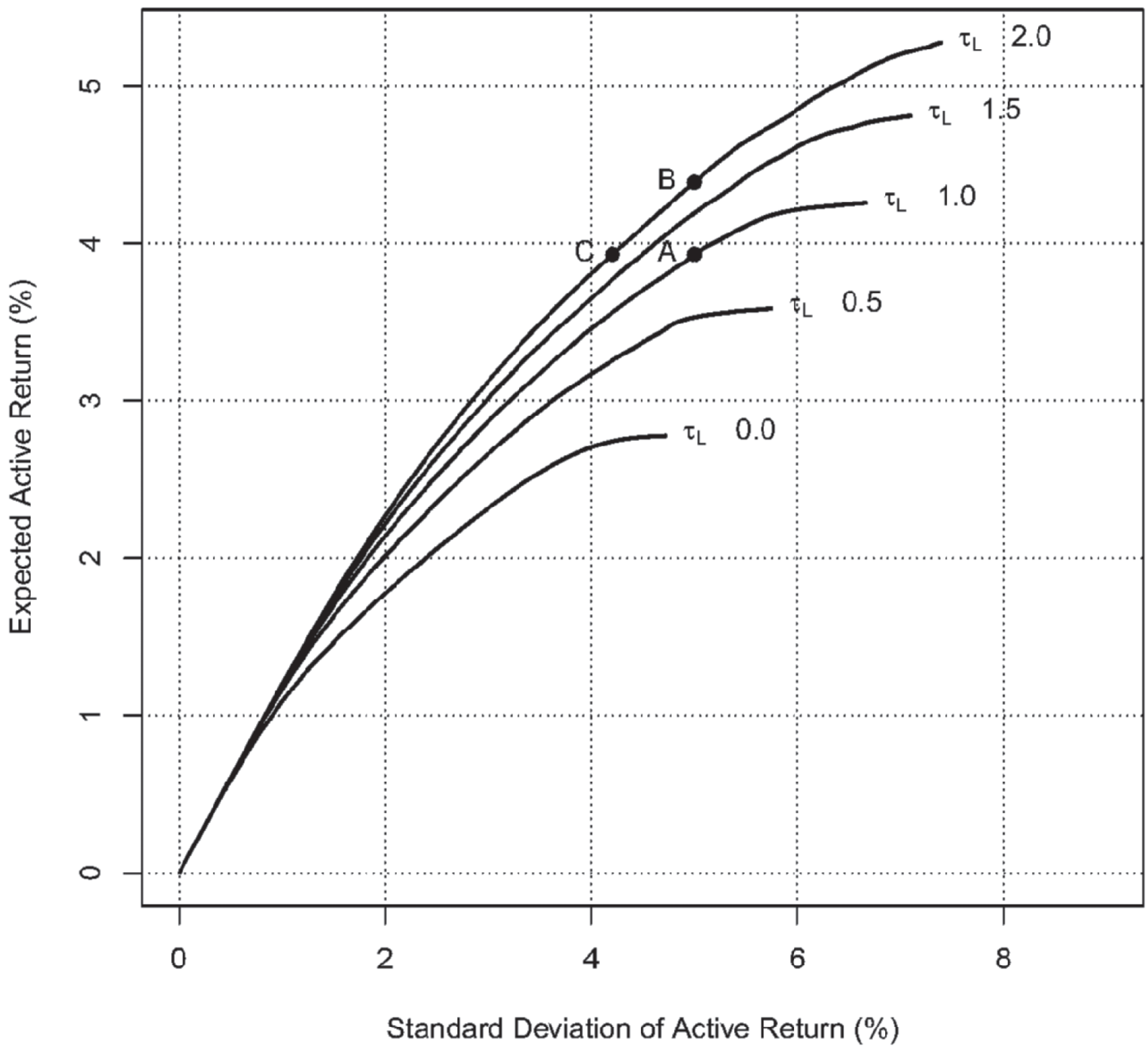
Because each portfolio is just a leveraged version of the same set of active positions, and we have assumed the EAE structure provides costless self-financing (i.e., short proceeds finance additional long positions), the efficient frontier is simply a straight line. In this case, ever-higher levels of leverage are used to achieve ever-higher expected returns, along with ever-higher standard deviations of return. As with the third panel, the investor derives the same efficient frontier by using conventional mean–variance optimization or mean–variance-leverage optimization, because no disutility is associated with the unique risks of leverage.

EFFICIENT FRONTIERS FOR VARIOUS LEVERAGE-TOLERANCE CASES

Exhibit 2 displays five different efficient frontiers on one chart. Each frontier corresponds to a different

EXHIBIT 2

Efficient Frontiers for Various Leverage-Tolerance (τ_L) Cases with the 10% Security Active Weight Constraint



level of leverage tolerance within a leverage-tolerance range of 0 to 2. Here 0 leverage tolerance again represents an investor who is unwilling to use leverage, and higher efficient frontiers correspond to investors with greater leverage tolerances. The frontier portfolios are constrained by the 10% active security weight constraint.

It might at first appear from the exhibit that the highest level of leverage tolerance results in the dominant efficient frontier; that is, higher leverage lets the investor achieve higher returns at any given level of volatility.[8] But one must consider the leverage-tolerance level associated with each efficient frontier. When leverage aversion is considered, it becomes apparent that each frontier consists of the set of optimal portfolios for an investor with the given level of leverage tolerance.

For example, consider the three portfolios represented by the points labeled A, B, and C in Exhibit 2. (We provide their characteristics in Exhibit 3.) Portfolio A is the optimal portfolio for investor A, who has a leverage tolerance of 1 and a volatility tolerance of 0.24. This is a 125-25 portfolio with a standard deviation of active return of 5% and an expected active return of about 3.93%. As shown in the last column, the utility for investor A, U_A, of portfolio A is 2.93. In other words, investor A is indifferent between portfolio A, which has an expected return of 3.93%, along with volatility and

EXHIBIT 3
Portfolio Characteristics

	τ_L	τ_V	EAE*	σ_P	α_P	U_A
A	1.00	0.24	125-25	5.00	3.93	2.93
B	2.00	0.14	139-39	5.00	4.39	2.72
C	2.00	0.09	135-35	4.21	3.93	2.68

**Rounded to the nearest percent.*

leverage risk, and a hypothetical portfolio with a certain return of 2.93% and no volatility and no leverage risk.

Portfolio B dominates portfolio A in an expected active return–standard deviation framework, because it offers a higher expected active return of 4.39% at the same 5% level of standard deviation of active return. But it is only optimal for an investor with a leverage tolerance of 2 and a volatility tolerance of 0.14; it is suboptimal for an investor with a leverage tolerance of 1. Portfolio B represents a 139–39 enhanced active portfolio; it entails significantly more leverage than the 125–25 portfolio A. The last column of Exhibit 3 shows that the utility of portfolio B for investor A is 2.72, which is lower than the 2.93 utility of portfolio A for investor A. The disutility of incurring the additional leverage risk more than offsets the benefit of the incremental expected return for this investor with less tolerance for leverage, and so portfolio B is suboptimal for investor A.

Finally, consider portfolio C, which has the same 3.93% expected active return as portfolio A. This is the optimal portfolio for investor C, who has a leverage tolerance of 2 and a volatility tolerance of 0.09. This portfolio also dominates portfolio A in an active return–standard deviation framework, because it offers the same expected return at a lower standard deviation of active return.

Although portfolio C is optimal for an investor with a leverage tolerance of 2, it is suboptimal for investor A, who has a leverage tolerance of 1, for the same reason that portfolio B is suboptimal: It entails more leverage than portfolio A, at 135–35 versus 125–25. Again, the disutility of the additional leverage risk more than offsets the benefit of the lower volatility for the investor with less tolerance for leverage. See this in the last column of Exhibit 3, which shows that investor A receives utility of 2.68 from portfolio C, lower than the 2.93 from portfolio A.

Exhibit 2 demonstrates that conventional mean–variance optimization and efficient frontier analysis are inadequate to determine optimal portfolios when investors use leverage but are averse to leverage risk. The conventional approach fails to recognize that most investors are willing to sacrifice some expected return in order to reduce leverage risk, just as they sacrifice some expected return in order to reduce volatility risk. Because the efficient frontier differs for investors with different tolerances for leverage, mean–variance–leverage optimization must be used to solve for optimal portfolios.

For each of the five efficient frontiers in Exhibit 2, volatility tolerance ranges from 0 (the origin) to 2 (the rightmost endpoint on each frontier). A curve connecting these endpoints would identify portfolios that are optimal for investors with a volatility tolerance of 2 and leverage tolerances ranging from 0 to 2. Which portfolio along this curve is optimal for a particular investor? The answer depends on the investor's leverage tolerance. (Note that, because different security active weight constraints become binding as one moves along each of the constant leverage-tolerance frontiers, a curve connecting the endpoints would not be smooth.)

An investor with a volatility tolerance of 0 and any level of leverage tolerance will choose the portfolio located at the origin: an index fund. An investor with a leverage tolerance of 0 will choose, from the lowest frontier shown, the portfolio consistent with the investor's volatility tolerance. An investor with a leverage tolerance of 2 will choose, from the highest frontier shown, the portfolio consistent with the investor's volatility tolerance. The optimal portfolio for an investor with any pair of leverage-tolerance and volatility-tolerance values between 0 and 2 will lie somewhere within the perimeter defined by the leverage- and volatility-tolerance frontiers of 0 and 2. Both volatility tolerance *and* leverage tolerance must be specified to determine the optimal portfolio for a given investor.

THE EFFICIENT REGION

With mean–variance–leverage optimization, optimal portfolios lie on a three-dimensional mean–variance–leverage surface. The choice of an optimal surface portfolio for a given investor depends on that investor's tolerances for volatility risk and leverage risk. Every leverage-tolerance level has a corresponding two-dimensional mean–variance efficient frontier. Similarly, for a particular level of volatility tolerance, there is a corresponding two-dimensional mean–variance efficient frontier.

EXHIBIT 4

Efficient Frontiers for Various Leverage- (τ_L) and Volatility- (τ_V) Tolerance Cases with No Security Active Weight Constraint

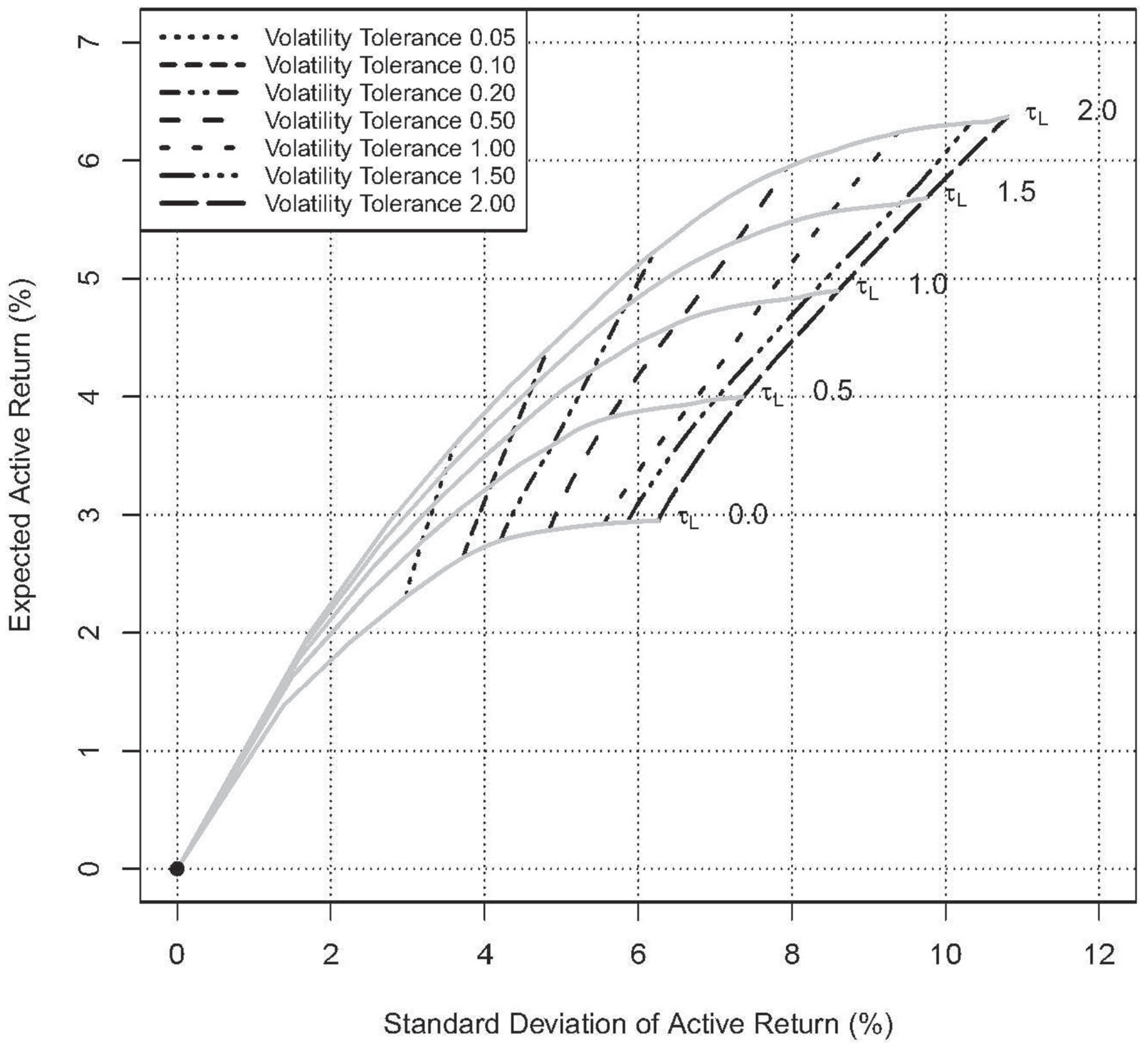


Exhibit 4 illustrates the efficient frontiers for various levels of investor leverage tolerance and those for various levels of investor volatility tolerance. Because Exhibit 4 assumes no constraint on the security active weights, the curve linking the optimal portfolios for an investor with a volatility tolerance of 2 is smooth (unlike in Exhibit 2).

Furthermore, without the security active weight constraints, both the standard deviation of active return and the expected active return range higher than in Exhibit 2. As either volatility tolerance or leverage tolerance declines from 2, the frontiers shift to the left and downward. When volatility tolerance is 0, the optimal portfolio—an index fund—lies at the origin. Depending on the investor's leverage and volatility tolerances, the optimal portfolio will lie somewhere in the mean–variance-leverage *efficient region* shown. Once again, the critical roles of both leverage and volatility tolerance in portfolio selection are apparent.

CONCLUSION

Conventional portfolio theory and mean–variance optimization must be augmented to incorporate leverage aversion. We propose that portfolio theory's mean–variance utility function include a term for leverage aversion, thereby transforming it into a mean–variance–leverage utility function. We use this specification to show the effects of leverage aversion on the efficient frontier.

Conventional mean–variance optimization considers only a portfolio's expected return and risk as measured by portfolio volatility. It implicitly assumes that investors have an infinite tolerance for the unique risks of leverage. In a mean–variance framework, investors prefer highly leveraged portfolios, because they offer the highest expected active return at each level of active risk.

Leverage, however, entails its own unique set of risks and costs. For instance, leverage can give rise to margin calls, which can force borrowers to liquidate securities at adverse prices due to illiquidity. It can result in losses exceeding the capital invested, and even in bankruptcy. Most investors are willing to sacrifice some expected return in order to reduce leverage risk, just as they sacrifice some expected return in order to reduce volatility risk. The highly leveraged portfolios that result from conventional mean–variance optimization entail too much leverage risk for leverage-averse investors.

We show that, when leverage aversion is included in portfolio optimization, lower mean–variance–leverage efficient frontiers having less leverage are optimal. Which frontier is optimal for a particular investor depends upon that investor's leverage tolerance. The optimal portfolio on that frontier for that investor depends upon that investor's volatility tolerance.

A mean–variance–leverage efficient region lies within bounded ranges of investor volatility tolerance and leverage tolerance. An investor's volatility and leverage tolerances determine the location of that investor's optimal portfolio within that region. Both volatility tolerance and leverage tolerance play critical roles in portfolio selection. Investor leverage aversion can have a large effect on portfolio choice.

APPENDIX

COMPARISON OF THE ENHANCEMENT SURFACES USING TWO DIFFERENT SPECIFICATIONS

As in Jacobs and Levy [2012], we chose 100 × 100 pairs of values for the tolerances (τ_V, τ_L) to cover the illustrative range [0.001, 2] for a total of 10,000 optimizations. Tolerances for volatility and leverage can be greater than 2, and as leverage tolerance approaches infinity, the optimal portfolio approaches that determined by a conventional mean–variance utility function.

To estimate the required inputs for Equation (6)—security expected active returns, covariances of security active returns, and covariances of security total returns—we used daily return data for the constituent stocks in the S&P 100 Index over the two years (505 trading days) ending on September 30, 2011. For estimating security expected active returns, we used a random transformation of actual active returns while maintaining a skill, or information coefficient (correlation between predicted and actual active returns), of 0.1, representing a manager with strong insight. For a description of the estimation procedure used, see Jacobs and Levy [2012]. We assumed the future covariances were known, so we calculated them based on the actual daily active returns and the actual daily total returns, respectively.

The results from this specification were broadly similar to the results from using the specification in Jacobs and Levy [2012], which used a constant based on an average of individual securities' active return variances, rather than the total variance of individual portfolios. At zero leverage tolerance, the optimal portfolios lie along the volatility-tolerance axis and have no leverage and hence no enhancement (they are long only). At zero volatility tolerance, the portfolios lie along the leverage-tolerance axis and have no active return volatility and hence hold benchmark weights in each security (as in an index fund). For portfolios above the axes, optimal enhancement is approximately independent of volatility tolerance if the latter is large enough. However, the optimal enhancement is highly dependent on the level of leverage tolerance chosen. This supports our assertion that investors should consider leverage tolerance when selecting an optimal portfolio.

The optimal enhancements using the new specification are slightly higher (by less than five percentage points) than those derived under the prior specification. This is not surprising, given the relationship between the two specifications. Note that the utility function represented by Equation (4) is equivalent to that of Equation (1) if one multiplies the leverage risk term of Equation (1) by the ratio:

$$R = \frac{\sigma_T^2}{c} \tag{A-1}$$

Using the expression for the variance of the portfolio's total return from Equation (5) and also Equation (3), Equation (A-1) can be rewritten as:

$$R = \frac{\sum_{i=1}^{N}\sum_{j=1}^{N} h_i q_{ij} h_j}{\frac{1}{N}\sum_{i=1}^{N}\omega_i^2} \tag{A-2}$$

This expression is the ratio of the portfolio's total return variance to the average (across all securities in the selection universe) of the variance of each stock's active returns. Calculating Equation (A-2) across the 10,000 optimal portfolios found by using the same constraint set for an enhanced active equity (EAE) portfolio and the same sample of S&P 100 stocks as in Jacobs and Levy [2012], we found $R \approx 0.85$.

As might be expected with a ratio close to 1, the results from optimization using Equation (4) were similar to those from using Equation (1). The major difference is that the new specification indicates that slightly more leverage is optimal than in the earlier specification, within the risk-tolerance ranges examined. This is because the ratio R is less than 1, implying a lower penalty for leverage risk in Equation (4) than in Equation (1).

It is difficult to draw general conclusions from this comparison, however, because R will vary, depending on portfolio structure, the particular portfolio and its level of enhancement, the sample data, and so on. The optimal portfolios in Jacobs and Levy [2012] derive from a constant that was estimated from the average of individual securities' active return variances. The results reflected in this article rely on the total return variance of a diversified portfolio. Because total return variance is larger than active return variance, this will raise R, while portfolio-diversification effects will lower R. The net effect depends on the particular situation, so R may be greater than or less than one.

ENDNOTES

We thank Judy Kimball and David Starer for helpful comments.

[1]In a section entitled "The Effect of Leverage," Kroll, Levy, and Markowitz [1984] stated: "Leverage increases the risk of the portfolio. If the investor borrows part of the funds invested in the risky portfolio, then the fluctuations of the return on these leveraged portfolios will be proportionately greater." In the present article, we consider other risks unique to using leverage.

[2]Certain legal entities, such as limited partnerships and corporations, can limit investors' losses to their capital in the entity. Losses in excess of capital would be borne by others, such as general partners who have unlimited liability or prime brokers.

[3]Incorporating a leverage-tolerance term in the utility function allows the investor to consider the economic trade-offs between expected return, volatility risk, and leverage risk. Using a constraint for leverage does not allow for consideration of the trade-offs with leverage risk. The level of leverage imposed with a constraint may be either too tight or too loose compared with the optimal leverage level, given the investor's leverage tolerance.

[4]The use of σ_p^2 as the measure of volatility risk is appropriate if active returns are normally distributed and the investor is averse to the variance of active returns, as well as for certain concave (risk-averse) utility functions (Levy and Markowitz [1979]). If the return distribution is not normal, displaying skewness or kurtosis ("fat tails") for instance, or the investor is averse to downside risk (semi-variance) or value at risk (VaR), the conclusions of this article still hold. That is, the investor should include a leverage-aversion term in the utility function, along with the appropriate measure of volatility risk, with neither risk term necessarily assuming normality.

Leverage may give rise to fatter tails in returns. For example, a drop in a stock's price may trigger margin calls, which may result in additional selling, while an increase in a stock's price may lead investors to cover short positions, which can make the stock's price rise even more.

Note that, if volatility risk is measured as the variance of total returns (such as for an absolute-return strategy) rather than as the variance of active returns, the conclusions of this article still hold.

[5]When the investor's leverage tolerance is 0, portfolio leverage, Λ, will be 0. Note that because short positions entail unlimited liability, like leveraged long positions, they expose the portfolio to losses beyond the invested capital. Hence, investors with 0 leverage tolerance would impose a non-negativity constraint on the h_is—that is, a no-shorting constraint.

We assume that investors have the same aversion to leveraged long positions as they do to short positions; however, this assumption may not be the case in practice, because short positions have potentially unlimited liability and are susceptible to short squeezes. One could model the aversion to long and short positions asymmetrically. Because doing so would complicate the algebra, for simplicity we use a common leverage tolerance.

[6]In practice there would be financing costs (such as stock loan fees); furthermore, hard-to-borrow stocks may entail higher fees. For more on EAE portfolios, see Jacobs and Levy [2007].

[7]For expository purposes, we estimate the variance of the portfolio's total return based on historical data in the same way that we estimate the variance of the portfolio's active return, but in practice an investor could estimate these variances on a forward-looking basis, taking into account

security position sizes relative to the market and the expected market impact upon liquidation. Note that leverage increases portfolio illiquidity. However, leverage and illiquidity are different, because illiquid portfolios without any leverage are not exposed to margin calls and cannot lose more than the capital invested.

[8]Note that the expected active returns shown do not reflect any costs associated with leverage-related events, such as forced liquidation at adverse prices or bankruptcy. These costs, however, are reflected in the disutility implied by the leverage-aversion term.

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